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| 2A.1A | The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations. |
| 2A.1B | The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments. |
| 2A.3A | The student is expected to analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems. |
| 2A.3B | The student is expected to use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities. |
| 2A.3C | The student is expected to interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts. |

Materials

Advanced Preparation:

- Student/teacher access to computers with TI Interactive and/or a graphing calculator and a projection device to use TI Interactive as a class demonstration tool.
- Chart paper
- Markers
- Student and group copies of handouts

For each student:

- Graphing calculator
- Student worksheets, *Casey's Part – Time Jobs, Selected Response Questions*
- Access to TI Interactive and TI Interactive Video Rental Sketches 1, 2, 3, and Video Rental Spreadsheet 1

For each student group of 3 to 4 students:

- Chart paper
- Markers
- Group worksheets *Andy and Beca, Video Joe, Shipping Costs*
- Access to TI Interactive and TI Interactive Video Rental Sketches 1, 2, 3, and Video Rental Spreadsheet 1

ENGAGE

The Engage portion of the lesson is designed to create student interest in the topic of video rentals, which will be explored later in the lesson as linear programming. This part of the lesson is designed for groups of three to four students.

1. Have students in groups of 3 or 4.
2. Distribute one copy of *Andy and Beca* to each group of students.
3. Working together, each group should complete the worksheet.
4. Each group will prepare a poster on their responses to Questions 12 – 14.
5. Post the chart papers around the room. Each group should choose one person from the group to stay with the group poster and explain their responses. Have the groups conduct a gallery walk to see all the other posters and solution strategies.
6. Come back together as a large group and debrief. Possible facilitation questions are shown below.

Facilitation Questions – Engage Phase

- How were the posters/solutions the same?
Answers may vary.
- How were the posters/solutions different?
Answers may vary.
- What were some of the strategies used?
Answers may vary. Some groups may have solved the problem graphically, some may have used a table containing a list of the possible outcomes, some may have used logical reasoning to solve the problems.
- Did you see any posters/solutions that included graphs? How were they used?
Answers may vary. If a student group used a graph to solve the problem, they likely have a shaded region bounded by the lines $x = 5$, $y = 7$, and $x + y = 10$.
- Did you see any posters/solutions that included tables? How were they used?
Answers may vary. If a student group used a table, they may have a 3-column table: number of VHS tapes, number of DVDs, and total cost.

EXPLORE

The Explore portion of the lesson provides the student with an opportunity to be actively involved in the exploration of the mathematical concepts addressed. This part of the lesson is designed for students to work in groups of 3 to 4 students.

1. Distribute copies of *Video Joe* to each group of students.
2. Have groups work through *Video Joe*.
3. When finished, each group should record their solutions on chart paper, answering all the questions included in the situation.
4. Have each group post their chart paper.

Facilitation Questions – Explore Phase

- How does the amount of shelf space limit the number of DVDs and VHS tapes Joe can stock?
He can only stock 1500 inches worth of videos.
- What other information in the problem limits the number of DVDs and VHS tapes he can stock?
Twice as many VHS as DVD, between 80 and 200 VHS
- What happened to the region as more inequalities, or restrictions, were applied?
The region became more limited.
- What does this limited region mean in the situation?
Each restriction in the situation limits the number of VHS tapes and DVDs that can be stocked.
- Do all the charts have the same feasible regions shown? How are they the same or different?
Graphs could have different windows/domain and range settings or different scaling.

EXPLAIN

The Explain portion of the lesson is directed by the teacher to allow the students to formalize their understanding of the TEKS addressed in the lesson.

1. Debrief the *Video Joe* activity. Use group posters when asking the Facilitation Questions.

Facilitation Questions – Explain Phase

- What is the inequality representing the number of VHS tapes and DVDs Video Joe can stock on his shelves? $5x + 4y \leq 1500$ Why is the inequality less than 1500 and not greater than 1500? *Video Joe can stock fewer videos, but not more videos that take up 1500 inches of shelf space. What do the '5' and '4' represent in the situation? each VHS tape takes up 4 inches of shelf space and each DVD takes up 5 inches of shelf space*
- Could you have graphed the inequality in standard form, by hand? *Yes, by using the x-intercepts and y-intercepts. What do the x-intercept and y-intercept represent in the situation? The x-intercept represents the number of DVDs Video Joe could stock if he had no VHS tapes and the y-intercept represents the number of VHS tapes Video Joe could stock if he had no DVDs.*
- What is the inequality in slope-intercept form? $y = -\frac{5}{4}x + 375$ Why is it important to put the inequality in slope-intercept form? *to plot the inequality quickly into TI Interactive or a graphing calculator. What does the y-intercept represent in the situation? the y-intercept represents the number of VHS tapes Video Joe could stock if he had no DVDs.*

Facilitation Questions – Explain Phase, continued

- What window settings did you use when graphing the inequality? Why? *Answers will vary. They should range from 0 – 400, approximately. It is important for them to understand why negative values are not necessary in this situation.*
- How did the inequality $y \leq 2x$ restrict the possible combinations of VHS tapes and DVDs Video Joe could stock? *Of the region represented by the first inequality, now the only combinations that can be used are the ones where the number of VHS tapes is at least twice the number of DVDs.*
- How did limiting the number of VHS tapes (between 80 and 200) restrict the possible combinations of VHS tapes and DVDs Video Joe could stock? *The number of possible combinations were decreased and limited to y-values between 80 and 200.*
- What happened to the original region as restrictions were added to the situation? *The original region got smaller each time a restriction was added.*
- The area common to all the restrictions of the situation is called the feasible region. Is the point (50 , 250) inside the feasible region? *No Explain your answer in terms of the situation. This point exceeds 200 VHS tapes, so therefore does not fit within the restriction that the number of VHS tapes has to be between 80 and 200.* Is the point (150 , 100) inside the feasible region? *Yes Explain your answer in terms of the situation. It meets all the restrictions of the situation.*
- Where is the point (100 , 200) in regards to the feasible region? *At the intersection of two of the inequalities.* How is this point different from the other two points you have looked at? *This point is located at an intersection of two inequalities, and therefore lies on the edge of the feasible region.* This point is one of the vertices of the feasible region. What are the coordinates of the other vertices? *(140 , 200), (236 , 80), (40 , 80).* What do all the vertices points have in common? *They all occur at the intersection of two of the restrictions of the feasible regions.* What are some methods that were used on the graphing calculator or in TI Interactive for finding the coordinates of the vertices? *Table, trace, calculate the intersection, graph*
- When calculating the amount of profit Video Joe could make, are there other combinations of VHS tapes and DVDs that would generate a different amount of profit (other than those listed in the table)? *Yes Do you think there are any of those combinations that would generate more than \$660 or less than \$240 in profit? No Why or why not? Use the spreadsheet to verify your answer. Students should enter various points in the spreadsheet trying to generate more than \$660 or less than \$240. By using the spreadsheet portion of TI Interactive, the students can readily see the profit being generated by the different combinations of VHS tapes and DVDs. The vertices of the feasible region contain the extremes of the region, therefore from those points you will find the maximum/minimum profit for Video Joe's store.*

ELABORATE

The Elaborate portion of the lesson provides an opportunity for the student to apply the concepts of the TEKS within a new situation. This part of the lesson is designed for students working in small groups.

1. Distribute **Shipping Costs** to each group of students. Divide students into an even number of groups.
2. All groups should have access to TI Interactive or a graphing calculator.
2. Have groups work together and record their actions on chart paper. Students should include all aspects of solving the problem on the chart paper and include sketches of the feasible region, inequalities, and justification of their solution using the cost function.
3. Put two groups together and have them explain to each other how they arrived at their solutions. After both groups have explained their strategies, the two groups should decide what their two posters have in common and how are they different.
4. Each pair of groups will present a summary to the whole class of the various strategies used, what their two posters had in common and how they were different. Use the Facilitation Questions to debrief the activity.

Facilitation Questions – Elaborate Phase

- How is this situation different from *Video Joe*?
This is asking for a minimum cost vs. maximum profit in Video Joe
- What were some of the things all the posters had in common?
Answers may vary.
- What were some of the differences between the posters?
Answers may vary.
- Describe the restrictions or limitations in this situation.
Truck capacity, capacity of each case, minimums/maximums of types of videos needed
- What were the window settings for the graphs? Why were these values chosen?
Answers may vary. Window settings should be chosen that are appropriate to the domain and range of the situation.
- Did you have to adjust the window settings of your graph as you added restrictions to the graph? Why?
Answers may vary. Depending on the original window chosen, sometimes adding restrictions will force students to zoom in on a certain region or to enlarge their window to see new vertex points.

EVALUATE

The Evaluate portion of the lesson provides the student with an opportunity to demonstrate his or her understanding of the TEKS addressed in the lesson.

1. Distribute *Casey's Part-time Jobs* to each student.
2. Each student should complete the assessment, showing all appropriate work.
3. Upon completion of the activity sheet, a rubric should be used to assess student understanding of the concepts addressed in the lesson.

Answers and Error Analysis for selected response questions:

<i>Question Number</i>	<i>TEKS</i>	<i>Correct Answer</i>	<i>Conceptual Error</i>	<i>Conceptual Error</i>	<i>Procedural Error</i>	<i>Procedural Error</i>	<i>Guess</i>
1	2A.1A	B	A	C	D		
2	2A.3A	A	C	D	B		
3	2A.3C	C	A	B			D
4	2A.3B	B	C	D	A		

Andy and Beca

Andy and Beca are renting videos for the weekend. They can only afford to rent a maximum of six videos. Some of the videos must be on VHS tapes and some must be on DVD.

1. What are the possible combinations of VHS and DVD Andy and Beca can rent? Use the table to list all the possible combinations.

VHS	DVD
1	1
1	2
1	3
1	4
1	5
2	1
2	2
2	3
2	4
3	1
3	2
3	3
4	1
4	2
5	1

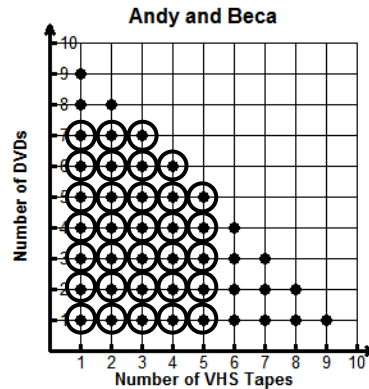
Open **Video Rental Sketch 1** through TI Interactive.

2. What does each plotted point represent on the graph?
Each point represents one possible combination of renting VHS tapes and DVDs.
3. Why are there no points with negative coordinates plotted on the graph?
You cannot rent a negative amount of tapes or DVDs.
4. Does your table match the table shown on the sketch?
Answers may vary.
5. Predict how the graph would change if Andy and Beca could rent a total of 10 videos.
There would be more combinations possible, therefore more points on the graph.
6. Open **Video Rental Sketch 2** to check your prediction. How do the two graphs compare? Explain.
Answers may vary.
7. What are the total number of combinations of renting VHS tapes and DVDs?
45 combinations

8. If Andy and Beca limit the number of VHS tapes to 5 or less and the number of DVDs to 7 or less, how would the graph change?

There would be 13 fewer points, or combinations of VHS tapes and DVDs.

9. Shade the graph below to show the new restrictions to the number of VHS tapes and DVDs Andy and Beca could rent.



10. What are the possible combinations, with the new restrictions included?

32 possible combinations

11. What are the outermost points of the restricted region?

(1, 7) (3, 7) (5, 5) (5, 1) (1, 1)

12. If VHS tapes rent for \$4 and DVDs rent for \$2, what is the most they could spend if they stay within all the restrictions? What combination of VHS tapes and DVDs would that be?

\$30 for 5 VHS tapes and 5 DVDs.

13. If VHS tapes rent for \$3 and DVDs rent for \$4, what is the most they could spend? What combination of VHS tapes and DVDs would that be?

\$37 for 3 VHS tapes and 7 DVDs.

14. How did the cost change from the first situation to the second situation? Why?

An increase of \$7; VHS tapes cost less and DVDs cost more in the second situation.

Video Joe

Video Joe has decided to open a small video rental store. He plans on offering DVDs and VHS tapes for rental. After installing all the shelves in the store, he calculates that he has 125 feet of shelf space to store the DVDs and VHS tapes. Each DVD takes up 5 inches of shelf space, while each VHS tape takes up 4 inches of shelf space.

Let x = the number of DVDs and y = the number of VHS tapes he can stock on his shelves at any given time.

1. Write an equation describing the number of VHS tapes and DVDs Video Joe can stock on his shelves, given the limited amount of shelf space.

$$5x + 4y = 1500 \quad 125 \text{ feet} = 1500 \text{ inches}$$

2. Would Video Joe be able to stock more or less VHS tapes and DVDs than represented by the equation? Justify your answer.
Less. With a defined amount of shelf space, Video Joe can always put fewer videos on the shelves, but it is impossible to put more.

3. Write the equation as an inequality to represent this situation.

$$5x + 4y \leq 1500$$

4. Write the inequality in slope – intercept form.

$$5x + 4y \leq 1500$$

$$4y \leq -5x + 1500$$

$$y \leq -\frac{5}{4}x + 375$$

5. Graph this inequality on TI Interactive or graphing calculator. Describe the region that would apply to this inequality.

The x-intercept is (300, 0) and the y-intercept is (0, 375). The region under the line applies to this inequality since Video Joe can use less than 1500 inches of shelf space, but not more.

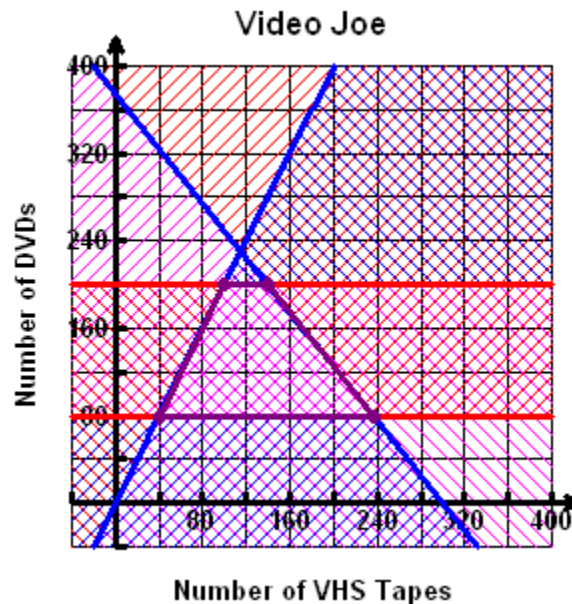
6. Video Joe would like to keep at most 2 times as many VHS tapes as DVDs. Write an inequality to represent this restriction.

$$y \leq 2x$$

7. Graph this inequality on TI Interactive or graphing calculator on the same screen as the previous inequality. Describe the region that now applies to the two restrictions (inequalities).

The common area that is located beneath each of the lines.

8. Video Joe would also like to keep between 80 and 200 VHS tapes in stock. Write two inequalities to represent this restriction.
 $y \geq 80$ or $80 \leq y \leq 200$
 $y \leq 200$
9. Graph these two inequalities on TI Interactive or graphing calculator on the same screen as the two previous inequalities. What region represents all the restrictions, or inequalities, in this situation?
The common area beneath the first two inequalities and between $y = 80$ and $y = 200$.
10. Open **Video Rental Sketch 3**. How does your graph compare to this one? Explain. (The purple trapezoidal region represents the region common to all restrictions.)
Answers may vary.



11. What are the vertices of the region common to all the restrictions (**feasible region**)?
 $(100, 200)$ $(140, 200)$ $(236, 80)$ $(40, 80)$
12. What do these coordinates represent in this situation?
 100 DVDs, 200 VHS tapes
 140 DVDs, 200 VHS tapes
 236 DVDs, 80 VHS tapes
 40 DVDs, 80 VHS tapes
13. Video Joe makes a profit of \$2.25 on each DVD rented and \$1.50 on each VHS tape rented. Write a function representing the profit he makes if he rents x number of DVDs and y number of VHS tapes.

$$f(x, y) = 2.25x + 1.50y$$

14. Use the profit function to determine the amount of profit Video Joe would make using the coordinates of the feasible region.

Students could use a calculator or the spreadsheet feature on TI Interactive to calculate each value.

15. Use the spreadsheet in TI Interactive to enter the coordinates of the feasible region and the profit function. Open **Video Rental Spreadsheet 1** to verify your answers. How do these answers compare with yours? Explain any differences.

Answers may vary.

DVDs	VHS	Profit
100	200	600
140	200	660
236	80	534
40	80	240

16. Which combination would generate the most profit for Video Joe, but still meet all the restrictions? How do you know?

140 DVDs and 200 VHS tapes will generate \$660 for Video Joe. Using a trial-and-error approach, students will not find another combination of DVDs and VHS tapes that will generate profit greater than \$660.

Shipping Costs

Video Joe orders all his DVDs and VHS tapes from an area supplier. The supplier has only one truck available for delivery and it has a capacity of 3600 cubic feet. One case of VHS tapes takes up 18 cubic feet of space, while one case of DVDs takes up 12 cubic feet of space. Video Joe places an order with the supplier for one truckload of VHS tapes and DVDs. He has to order between 150 and 240 cases of DVDs to meet the demand and at least 20 cases of VHS tapes. The shipping costs are based on the number of cases on the truck. Each case of VHS tapes costs \$3.50 in shipping costs and each case of DVDs costs \$3.75 in shipping costs.

Let x = number of cases of VHS tapes
 y = number of cases of DVDs

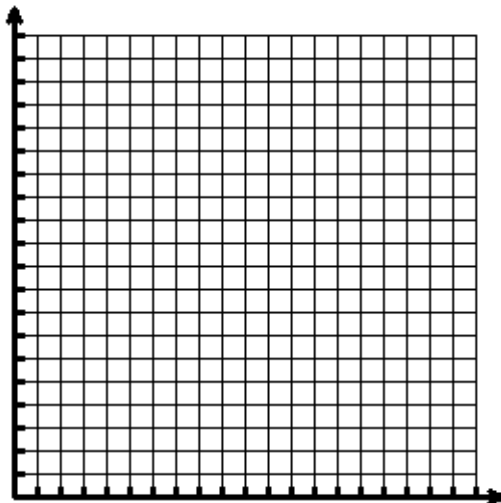
How many cases of VHS tapes and DVDs should he order if he would like to pay the least amount possible in shipping costs and stay within all the restrictions?
20 cases of VHS tapes and 150 cases of DVDs

Casey's Part-time Jobs

Casey the college student is working two part-time jobs. He works at a video rental store for \$5.25 per hour and at a movie theatre for \$6.05 per hour. He wants to work no more than 30 hours per week. He wants to work between two and three times the hours at the movie theatre than at the video store. He also has to work a minimum of 10 hours per week at the movie theatre.

Let x = the number of hours worked at the video store
 y = the number of hours worked at the movie theatre

Use TI-Interactive or a graphing calculator to graph the feasible region described above. Record the feasible region below. Label the axes and the vertices of the feasible region.



How many hours should he work at each job to earn the maximum amount of money each week?

What is the maximum amount of money he could make each week? Justify your answers.

He should work 10 hours a week at the video store and 20 hours a week at the movie theatre.

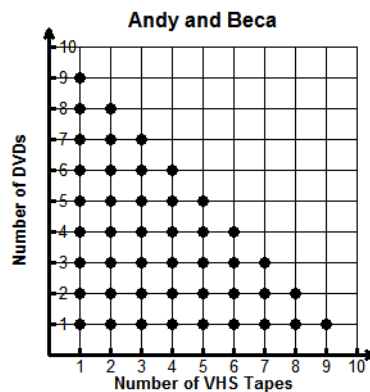
\$173.50

6. Open **Video Rental Sketch 2** to check your prediction. How do the two graphs compare? Explain.

7. What are the total number of combinations of renting VHS tapes and DVDs?

8. If Andy and Beca limit the number of VHS tapes to 5 or less and the number of DVDs to 7 or less, how would the graph change?

9. Shade the graph below to show the new restrictions to the number of VHS tapes and DVDs Andy and Beca could rent.



10. What are the possible combinations, with the new restrictions included?

11. What are the outermost points of the restricted region?

12. If VHS tapes rent for \$4 and DVDs rent for \$2, what is the most they could spend if they stay within all the restrictions? What combination of VHS tapes and DVDs would that be?
13. If VHS tapes rent for \$3 and DVDs rent for \$4, what is the most they could spend? What combination of VHS tapes and DVDs would that be?
14. How did the cost change from the first situation to the second situation? Why?

Video Joe

Video Joe has decided to open a small video rental store. He plans on offering DVDs and VHS tapes for rental. After installing all the shelves in the store, he calculates that he has 125 feet of shelf space to store the DVDs and VHS tapes. Each DVD takes up 5 inches of shelf space, while each VHS tape takes up 4 inches of shelf space.

Let x = the number of DVDs and y = the number of VHS tapes he can stock on his shelves at any given time.

1. Write an equation describing the number of VHS tapes and DVDs Video Joe can stock on his shelves, given the limited amount of shelf space.
2. Would Video Joe be able to stock more or less VHS tapes and DVDs than represented by the equation? Justify your answer.
3. Write the equation as an inequality to represent this situation.
4. Write the inequality in slope – intercept form.
5. Graph this inequality on TI Interactive or graphing calculator. Describe the region that would apply to this inequality.
6. Video Joe would like to keep at most 2 times as many VHS tapes as DVDs. Write an inequality to represent this restriction.

7. Graph this inequality on TI Interactive or graphing calculator on the same screen as the previous inequality. Describe the region that now applies to the two restrictions (inequalities).

8. Video Joe would also like to keep between 80 and 200 VHS tapes in stock. Write two inequalities to represent this restriction.

9. Graph these two inequalities on TI Interactive or graphing calculator on the same screen as the two previous inequalities. What region represents all the restrictions, or inequalities, in this situation?

10. Open **Video Rental Sketch 3**. How does your graph compare to this one? Explain. (The purple trapezoidal region represents the region common to all restrictions.)

11. What are the vertices of the region common to all the restrictions (**feasible region**)?

12. What do these coordinates represent in this situation?

13. Video Joe makes a profit of \$2.25 on each DVD rented and \$1.50 on each VHS tape rented. Write a function representing the profit he makes if he rents x number of DVDs and y number of VHS tapes.
14. Use the profit function to determine the amount of profit Video Joe would make using the coordinates of the feasible region.
15. Use the spreadsheet in TI Interactive to enter the coordinates of the feasible region and the profit function. Open **Video Rental Spreadsheet 1** to verify your answers. How do these answers compare with yours? Explain any differences.
16. Which combination would generate the most profit for Video Joe, but still meet all the restrictions? How do you know?

Shipping Costs

Video Joe orders all his DVDs and VHS tapes from an area supplier. The supplier has only one truck available for delivery and it has a capacity of 3600 cubic feet. One case of VHS tapes takes up 18 cubic feet of space, while one case of DVDs takes up 12 cubic feet of space. Video Joe places an order with the supplier for one truckload of VHS tapes and DVDs. He has to order between 150 and 240 cases of DVDs to meet the demand and at least 20 cases of VHS tapes. The shipping costs are based on the number of cases on the truck. Each case of VHS tapes costs \$3.50 in shipping costs and each case of DVDs costs \$3.75 in shipping costs.

Let x = number of cases of VHS tapes
 y = number of cases of DVDs

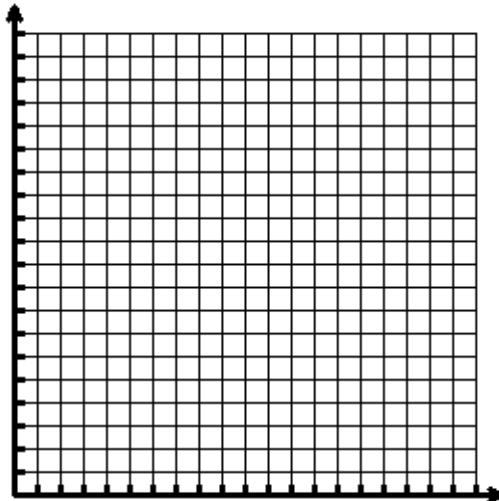
How many cases of VHS tapes and DVDs should he order if he would like to pay the least amount possible in shipping costs and stay within all the restrictions?

Casey's Part-time Jobs

Casey the college student is working two part-time jobs. He works at a video rental store for \$5.25 per hour and at a movie theatre for \$6.05 per hour. He wants to work no more than 30 hours per week. He wants to work between two and three times the hours at the movie theatre than at the video store. He also has to work a minimum of 10 hours per week at the movie theatre.

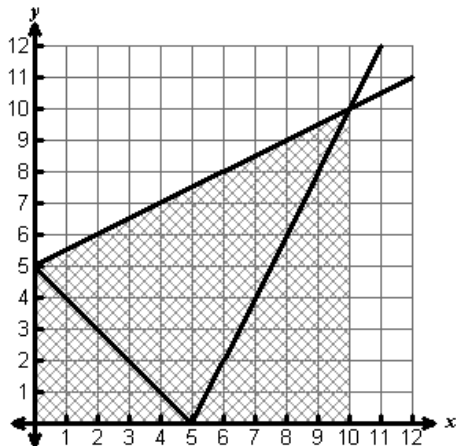
Let x = the number of hours worked at the video store
 y = the number of hours worked at the movie theatre

Use TI-Interactive or a graphing calculator to graph the feasible region described above. Record the feasible region below. Label the axes and the vertices of the feasible region.



How many hours should he work at each job to earn the maximum amount of money each week? What is the maximum amount of money he could make each week? Justify your answers.

- 1 Shown below is a feasible region. The profit function for the region is $f(x, y) = 6x + 5y$.



What are the minimum and maximum values of the function?

- A 5 and 10
- B 25 and 110
- C 25 and 30
- D 0 and 110

- 2 A company machines and sells nuts and bolts. One machine can produce 100 nuts and 50 bolts each hour. Due to demand, the company must produce at least two times more nuts than bolts. The machine runs a maximum of 40 hours per week. Which set of inequalities shows the correct restrictions for this situation if x represents the number of nuts produced in one week and y represents the number of bolts produced in one week?

A
$$\frac{x}{100} + \frac{y}{50} \leq 40$$

$$y \geq \frac{1}{2}x$$

B
$$\frac{x}{50} + \frac{y}{100} \leq 40$$

$$y \geq \frac{1}{2}x$$

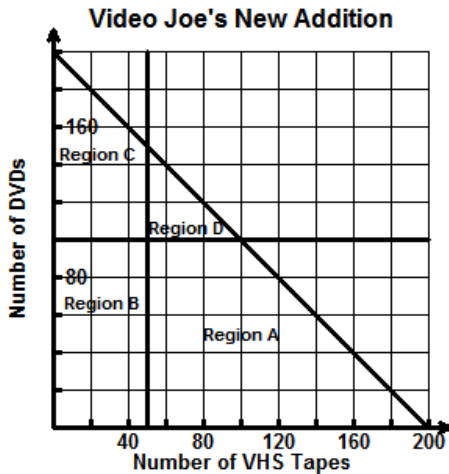
C
$$\frac{x}{100} + \frac{y}{50} \leq 40$$

$$y \geq 2x$$

D
$$\frac{x}{50} + \frac{y}{100} \leq 40$$

$$y \geq 2x$$

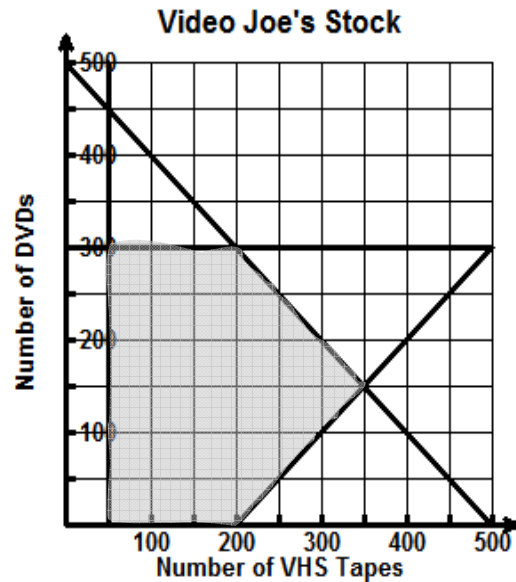
- 3 Video Joe is expanding his video store. He added enough shelving to hold a maximum of 200 items. He wants to have 50 VHS tapes at most and at least 100 DVDs in stock at all times in the new addition.



Which of the following regions represents the limited restrictions of this situation?

- A Region A
- B Region B
- C Region C
- D Region D

- 4 The feasible region shown below represents the possible amounts of VHS tapes and DVDs on Video Joe's shelves at any given time.



If he makes \$1.75 on each VHS rental and \$2.00 on each DVD rental, which combination of VHS tapes and DVD rentals would result in the most profit?

- A 350 VHS and 150 DVD
- B 200 VHS and 300 DVD
- C 150 VHS and 350 DVD
- D 300 VHS and 200 DVD

- 2A.1(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.
- 2A.4(B) extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions
- 2A.8(B) analyze and interpret the solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula
- 2A.11(B) use the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describe limitations on the domains and ranges, and examine asymptotic behavior
- 2A.11(C) determine the reasonable domain and range values of exponential and logarithmic functions, as well as interpret and determine the reasonableness of solutions to exponential and logarithmic equations and inequalities
- 2A.11(F) analyze a situation modeled by an exponential function, formulate an equation or inequality, and solve the problem.
- G.1(B) recognize the historical development of geometric systems and know mathematics is developed for a variety of purposes
- G.2(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships
- G.5(A) use numeric and geometric patterns to develop algebraic expressions representing geometric properties
- G.5(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;
- G.11(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures
- G.11(B) use ratios to solve problems involving similar figures

Materials

Advanced Preparation:

- Student access to computers with Geometer's Sketchpad and necessary sketches and/or a projection device to use the computer activities as a demonstration
- Student or teacher access to the Internet
- Chart paper and markers or blank transparencies
- Graphing calculator connected to an overhead projector or presenter
- Transparencies provided with this lesson
- Copies of "Technology Tutorial: The Golden Ratio" for each group of students.

For each student:

- Graphing calculator
- **The Eye of the Beholder** activity sheet
- **Creating a “Golden” Exponential Function** activity sheet
- **Algebra and the Golden Ratio** activity sheet
- **The Golden Ratio in Art and Architecture** activity sheet
- **Golden Areas** activity sheet

ENGAGE

The Engage portion of the lesson is designed to create student interest in the application of the number phi (golden ratio) to everyday life. This part of the lesson is designed for groups of two to four students using a computer station for Internet access and The Geometer’s Sketchpad.

1. Show **Transparency – The Eye of the Beholder**.
2. Hand each student a copy of the activity sheet **The Eye of the Beholder** and have each student group open the Geometer’s Sketchpad sketch **Face Sample**. Students will use Geometer’s Sketchpad to obtain data to calculate ratios of the facial features indicated on the sketch. They will record these ratios on the activity sheet. Each ratio will be approximately 1.6.
3. Groups will also log on to the Internet and use the website shown on their activity sheet or similar sites that have photos of celebrities. Each group is to find one celebrity photo that they want to copy and insert into Geometer’s Sketchpad. If your school’s firewall blocks the use of such sites, you can download some photos yourself and give them to the students in a Word file. They can then copy and insert whichever photo they choose.
4. Students may want to open a new sketch in which to insert their celebrity photo instead of inserting into the same file as the sample. Once students have inserted a copy of the celebrity photo, they will use Geometer’s Sketchpad and take the same types of measurements as on the face sample and record these on the activity sheet.
5. Once the groups have recorded their measurements and ratio values, have them share their findings with the whole class. You may want to record their findings on chart paper or a transparency so students can have fun discussing who is the most handsome or beautiful.

Facilitation Questions – Engage Phase

- What facial features do you find attractive in another person?
Answers may vary. Let students briefly share to enhance the engagement aspect.
- Besides ratios, what other mathematical terms or concepts can you apply to a person's physical appearance?
Symmetry, orientation of features (crooked smile, etc.), overall size of head compared to body
- Do certain feature measurements of species other than humans have an affect on their "attractiveness"?
Animals look for features in their potential mates such as overall proportion, strength, speed, etc. to ensure that their offspring have a greater chance of survival. Judges at dog and cat shows look for certain physical features to judge the pureness of a breed.
- What are some of the different ways that ratios can be expressed?
Ratios can be written as fractions or decimals or using the word "to" or a colon to separate the terms of the ratio.
- What is the decimal value (to the nearest tenth) of the ratios found from the face sample?
They all were close to 1.6.
- Consider the following ratios. What are their decimal equivalents and what do you notice? $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, $\frac{21}{13}$
Each new numerator is equal to the sum of the numerator and denominator from the previous fraction; the decimal values seem to approach about 1.6.
- How do the decimal equivalents in the sequence of ratios above compare to the ratios in your celebrity's facial feature ratios?
In the most attractive faces, the ratios were close to 1.6 – same as the ratios in the sequence.

EXPLORE

The Explore portion of the lesson provides an opportunity for the student to connect the concept of the golden ratio to an exponential function. This part of the lesson is designed for groups of two to four students working with Geometer's Sketchpad and a graphing calculator.

1. Distribute the **Creating a "Golden" Exponential Function** activity sheet.
2. Inform students that there are several geometric models of the golden ratio. A representation of the exact value of the golden ratio will be discovered through patterns of golden triangles. Display **Transparency 1 – The Golden Triangle**. On the transparency, sketch a bisector of $\angle BAC$ and name its intersection with the leg \overline{BC} point D.

3. Prompt students to recognize characteristics of $\triangle ABC$ and $\triangle CAD$ that they may remember from Geometry such as:

Facilitation Questions – Explore Phase

- What are the measures of the base angles of $\triangle ABC$? Why?
72°. If the vertex angle is 36°, that leaves 180-36 = 144 degrees to split equally among the two base angles.
- If you bisect $\angle BAC$, what are the measures of the two new angles? Why?
36°, because bisecting an angle cuts the angle measure in half.
- How are $\triangle ABC$ and $\triangle CAD$ related?
They are both isosceles triangles with 36° vertex angles.
- Are $\triangle ABC$ and $\triangle CAD$ similar? How do you know?
Yes, because all corresponding angles are congruent.
- What does this imply about the lengths of certain sides of $\triangle ABC$ and $\triangle CAD$? Why?
Corresponding sides have proportional lengths, such as $\frac{BC}{AC} = \frac{AD}{DC}$. Other proportions can be made using corresponding sides in a similar fashion.

4. Students should open the Geometer's Sketchpad sketch **Golden Triangle** and find the measurements and ratios indicated. To find the measurements, students can click on the appropriate action buttons.
5. Prompt students to enter their measurements on the activity sheet. Each group will probably have different segment measurements if they resized the triangle in the sketch. However, the ratios should all be the same, about 1.6.
6. Use **Transparency 2 – The Golden Triangle** to show how to create more triangles within the original triangle. As more triangles are created, help students see each new golden ratio. Use colored markers if possible. The **Golden Triangle 2** page in the Geometer's Sketchpad sketch **Golden Triangle** has the triangles already created, but students will have to find the measurements on their own.
7. Students could spend lots of time listing every ratio from the seven nested triangles. However, our emphasis now is to show how the golden triangles are connected to an exponential function using 1.6 as the common ratio (e.g., "b" in the function $y = a \cdot b^x$.) The value of "a" would be whatever the student's initial leg length is. Do NOT encourage them to use the exponential regression. If you do, make sure you have a discussion about how the values are related to the data.
8. Important! Have students share the function that they derived and how they calculated it.

Facilitation Questions – Explore Phase

- What happens if you change the size of your triangle in either of the sketches?
The side lengths will change but the ratios all remain about 1.618.
- How many proportions can you make from the segments in **Golden Triangle 2**?
Answers may vary but there's a lot.
- Why are you asked to write an exponential function instead of a linear, quadratic, or other type of function?
From y-value to y-value, there is a common ratio which is about 1.618.
- What window did you use to display your data? Why?
The window depends on the size of the largest segment recorded, \overline{BC} . A possible window could be $x: 0, 10, 1$ and $y: 0, 10, 1$.
- How did you use your table to develop an exponential function for your data?
Answers may vary. Students may have found successive ratios of leg lengths in the table. They may have guessed at the initial value by using transformations of the graph.
- Is your function exactly the same as other students' functions? Why or why not?
No, the "b" value is the same (1.618) but the "a" value may be different. If the original triangle is changed in size, the side lengths will vary. However, all of the triangles are "golden," so the ratio of leg to base will always equal phi and is the ratio of the exponential function.
- How can you use the table, graph, and/or function to find the next term in the sequence of leg lengths?
Table: multiply the preceding term by 1.618
Graph: trace on the function curve to $x = 8$ and read the y-value
Function: on the home screen, enter $x = 8$ into the function rule and evaluate
- Where would the 8th term value appear on the set of golden triangles?
Extend side \overline{AC} out to the left. Construct a 36° angle with vertex B so that one side contains \overline{BA} and the other side intersects \overline{AC} . Label this point of intersection point P . The length of \overline{BP} is the 8th term. Its value should equal $BC \div 1.618$.
- If you created another triangle inside of $\triangle GHC$, describe the side that fits the data in your table.
Bisect $\angle GHC$ and name its intersection with \overline{GC} point J . The length of side \overline{HJ} has a value approximately equal to $GC \div 1.618$. This value would fit before the first term and equals the initial value used in the function.

EXPLAIN

The Explain portion of the lesson is directed by the teacher to allow the students to formalize their understanding of the actual value of the golden ratio, known as *phi*.

1. Refer to **Transparency – The Golden Section** to connect the golden ratio to the Fibonacci sequence. Revisit the proportion of the golden ratio on **Transparency – The Algebra of the Golden Ratio** in order to derive the exact value of the ratio.
2. Lead students through the discussion of solving the proportion. You may want to stay with the variable “a” when solving, then introduce the symbol for *phi* at the end.
3. Give students time to solve the quadratic equation and assist as needed. They should be able to come up with the exact value $\frac{1+\sqrt{5}}{2}$ or at least a good decimal approximation of about 1.61803.

Sample solution using the quadratic formula:

$$\Phi^2 = \Phi + 1$$

$$\Phi^2 - \Phi - 1 = 0$$

$$\text{Let } a = 1, b = -1, c = -1$$

$$\Phi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\Phi = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\Phi = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1-\sqrt{5}}{2} < 0, \text{ so it is an extraneous solution and } \Phi = \frac{1+\sqrt{5}}{2}.$$

4. Students will connect the Fibonacci sequence to another exponential function using the activity sheet, **Algebra and the Golden Ratio**. Give students time to work on the activity sheet with a partner then discuss their observations and results.

Facilitation Questions – Explain Phase

- Is the value of the golden ratio really a fraction/ratio?
*Technically, a fraction or rational number is a ratio of whole numbers so the "golden ratio" is **not** a fraction.*
- Are the ratios made from consecutive Fibonacci numbers the same as the golden ratio?
These ratios are approximations. The larger the Fibonacci numbers used, the better the approximation to phi. (Actually the limit of these ratios = the golden ratio but that may be a discussion for another class!)
- How did you solve the quadratic equation?
Answers may vary. Students could use the quadratic formula or find a reasonable approximation on a graphing calculator.
- Is there just one value for the golden ratio?
Yes and no. When you solve the quadratic, you get two answers, but the larger value of 1.61803... is commonly accepted as the value of the golden ratio, often called ϕ . Curiously however, the other value, 0.61803... shares the same decimal part and is equal to $\frac{1}{\phi}$.

ELABORATE

The Elaborate portion of the lesson provides the student with an opportunity to extend what they've learned to real-world applications in art and architecture. This part of the lesson is designed for students to work in groups of two to four.

1. Show how the numbers in the Fibonacci sequence approximate phi in the golden rectangle model. See **Transparency – The Golden Rectangle**.
2. Students will search the Internet for "golden ratio" and "art" or "architecture."
3. Give each group a copy of the activity sheet **The Golden Ratio in Art and Architecture**.
4. Students are to find one example of how the golden ratio has been used in architecture and one example of art (painting, sculpture, etc.). Each group will record their findings on the activity sheet and present their findings to the class.

Facilitation Questions – Elaborate Phase

- What examples did you find of the golden ratio?
Answers will vary but will probably include structures such as the pyramids, the Parthenon, the United Nations building, and the Notre Dame cathedral. Art may include works by Leonardo da Vinci, Georges Seurat, Rembrandt, and Salvador Dali.

Facilitation Questions – Elaborate Phase

- What are some other names for the golden ratio?
Golden mean, golden section, phi, tau (uncommon), divine proportion
- Did you run across any other examples of the golden ratio?
Answers may vary. In nature, spirals in flower petals, seed heads, pine cones, and leaves on stalks come in Fibonacci numbers. Shells such as the nautilus shell is formed in a spiral that illustrates the golden ratio.
- Did you come across any other models of the golden ratio other than the golden rectangle?
Probably so. Many websites show the golden ratio as it relates to a pentagon, a pentagram, a decagon, and a golden triangle.
- Did you find any symbols or notations that were new to you?
The symbol for phi, ϕ , is foreign to most students and is the only one we want to address in this lesson.

EVALUATE

The Evaluate portion of the lesson provides the student with an opportunity to demonstrate his or her understanding of the TEKS addressed in the lesson. This assessment is intended for groups of two to four students.

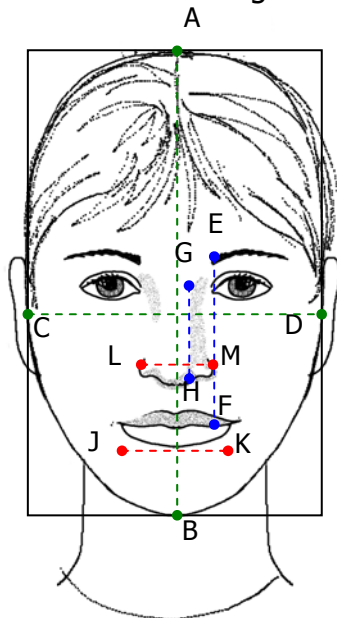
1. Provide each group a copy of the activity sheet **Golden Areas**.
2. Provide each student with a graphing calculator.
3. Upon completion of the activity sheet, a rubric should be used to assess student understanding of the concepts addressed in the lesson.

Answers and Error Analysis for selected response questions:

Question Number	TEKS	Correct Answer	Conceptual Error	Conceptual Error	Procedural Error	Procedural Error	Guess
1	2A.11F	A	B	D			C
2	2A.11B	B	A		D		C
3	2A.8B	C	A	B	D		
4	2A.1B	C	B		D		A

The Eye of the Beholder – Answer Key

- Study the features on the artist’s sketch below. Identify the segments that represent each of the following ratios.



Sketch by artist, Debra L. Hayden, 2005.
Used by permission

Ratio	Segments
Length of face, to Width of face	$\frac{\text{Length of face}}{\text{Width of face}} = \frac{AB}{CD} = 1.63$
Lips to eyebrows to Length of nose	$\frac{\text{Lips to eyebrows}}{\text{Length of nose}} = \frac{EF}{GH} = 1.61$
Width of mouth to Width of nose	$\frac{\text{Width of mouth}}{\text{Width of nose}} = \frac{JK}{LM} = 1.60$
Average Ratio	1.61

- Open the sketch **Face Sample** in Geometer’s Sketchpad. Calculate the ratio values indicated in the sketch by clicking on the “Measure Ratio” action button for the given ratio. Also, calculate the average ratio. Record your answers in your table.
- Log on to the Internet and open the website <http://www.angelfire.com/celeb2/celebrityfaces/>. Search for a photo of your favorite celebrity. The photo must be a full front view of the face.
- Right click on the face and select “Copy” so that you can “insert” the photo into Geometer’s Sketchpad.
- Using Geometer’s Sketchpad, construct and measure segments of the face you copied as shown on the sample. Measure the appropriate ratios and record them in the chart below. See “Technology Tutorial: The Golden Ratio” for assistance with the technology.

I used a photo of : *Brad Pitt*

Length of face	5.02 cm	Ratio	1.65
Width of face	3.05 cm		
Lips to eyebrows	1.77 cm	Ratio	1.62
Length of nose	1.09 cm		
Width of mouth	1.20 cm	Ratio	1.39
Width of nose	0.86 cm		

Answers will vary depending on photo chosen. However, ratios will probably be between 1.4 and 1.8.

6. How do your ratios compare with those found by other groups in the class? Why do you think this is so?

Answers may vary. Ratios should all be similar and fairly close to 1.61.

Creating a "Golden" Exponential Function

Answer Key

- Open the sketch **golden triangle1** to find possible measurements for each of the following:

Answers will vary.

The length of \overline{BC} = 9.6776

The length of \overline{AC} = 5.9811

The ratio of $\overline{BC} : \overline{AC} \approx$ 1.62

Answers will vary.

The length of \overline{BD} = 5.9811

The length of \overline{DC} = 3.6965

The ratio of $\overline{BD} : \overline{DC} \approx$ 1.62

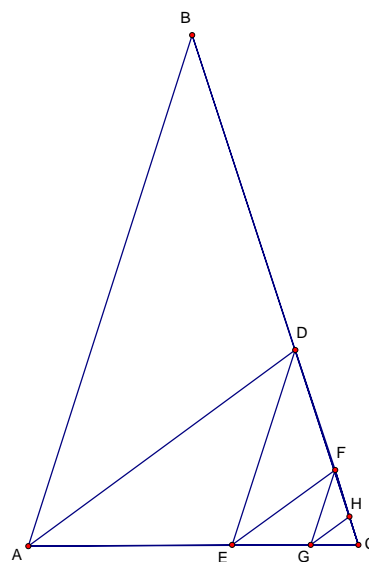
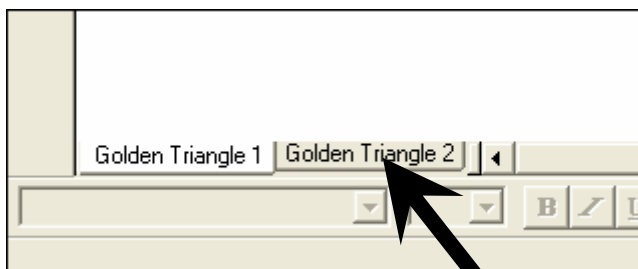
Use the Measure menu to find and display the lengths of segments BC, AC, BD, and DC.

- Measure Segment BC
- Measure Segment AC
- Measure Segment BD
- Measure Segment DC

Measure and display the ratios: $\frac{BC}{AC}$ and $\frac{BD}{DC}$.

- Find Ratio of BC to AC
- Find Ratio of BD to DC

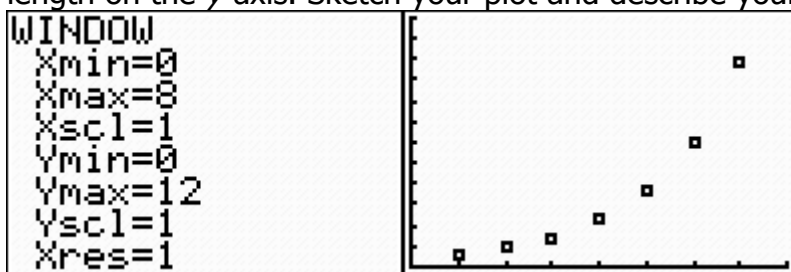
- Click on Point *C* and drag it around the screen. What happens to the segment lengths?
As the triangle gets bigger, the lengths increase. As the triangle gets smaller, the lengths decrease.
- What happens to the ratios when you drag point *C* around the screen?
The ratios stay the same, no matter how big the triangle gets.
- Click the **Golden Triangle 2** tab inside your sketch. Find possible values for the following:



Triangle	Leg	Length	Successive Ratios
1	\overline{HC}	0.546	
2	\overline{GC}	0.883	1.62
3	\overline{FC}	1.429	1.62
4	\overline{EC}	2.312	1.62
5	\overline{DC}	3.741	1.62
6	\overline{AC}	6.053	1.62
7	\overline{BC}	9.794	1.62

Answers will vary

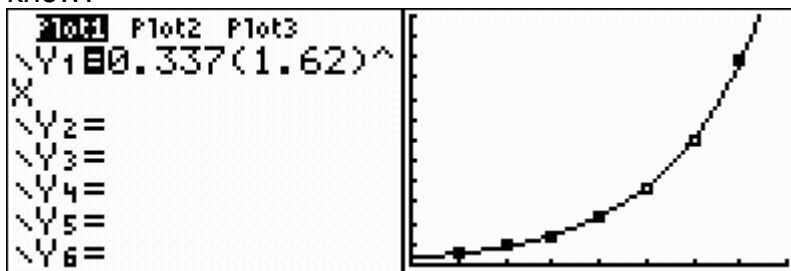
5. Enter the numbers 1 – 7 into List 1 on your graphing calculator. Enter the lengths of the segments \overline{HC} , \overline{GC} , \overline{FC} , \overline{EC} , \overline{DC} , \overline{AC} , and \overline{BC} into List 2. Create a scatter plot on your graphing calculator with the Triangle number on the x -axis and the Leg length on the y -axis. Sketch your plot and describe your window.



6. Determine an exponential function that passes through these points. Explain how you determined the function.

Answers may vary. By finding the successive ratios, the exponential function $y = 0.337(1.62)^x$ can be generated.

7. Sketch your plot and function graph. Does the function fit the data well? How do you know?



8. What does the coefficient in your function represent in the golden triangle? How did you obtain this value?

The coefficient represents the leg length of the "0" triangle, or the one preceding triangle GHC in the sequence. In other words, the coefficient is the initial value when $x = 0$. I had to divide the first leg length by 1.62 to find the y -value that corresponds with $x = 0$.

9. What does the base of the power in your function represent in the golden triangle?

The base represents the successive ratio between consecutive terms. In this case, it is the golden ratio rounded to approximately 1.62.

Algebra and the Golden Ratio

Answer Key

You have found the exact value of the golden ratio to be $\frac{1+\sqrt{5}}{2}$. Let's look at how this value connects to the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...

Consider the table below but don't fill in the right-hand column until you've answered questions 1 – 3.

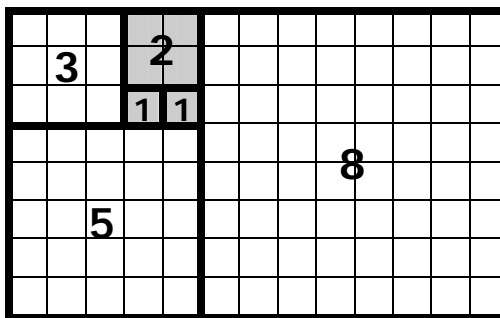
Term number	Fibonacci number
	5
1	8
2	13
3	21
4	34
5	55
6	89
7	144
8	233
9	377
10	610

- If you made a scatter plot of Fibonacci number vs. term number, what would the scatter plot look like?
It would look like an exponential curve except the first few points are off.
- If you started with 1 as your first Fibonacci number, could you write a function that would pass through all of the points in your scatter plot?
Starting with $y=1$ means that the first few points don't fit very well.
- How could you make a scatter plot that more closely fits an exponential function?
Shift to about the 4th or 5th Fibonacci number. I started with 8 and the graph was easier to fit.
- Fill in the table with the Fibonacci numbers of your choice and write an exponential function to fit your points.
 $y = 5 * 1.6^x$
- Which would give a better fit: starting with 5 or starting with 13? How does choosing a different starting number affect your function rule?
The further down the Fibonacci sequence you go, the closer the ratios of consecutive terms are to 1.6, so starting with 13 would be better than if you start with 5.

Golden Areas

Answer Key

Consider the squares that make up a golden rectangle shown below. The squares have sides that are terms of the Fibonacci sequence: 1, 1, 2, 3, 5, ... Each golden rectangle, such as the square that is shaded, is formed by attaching the next Fibonacci square to the previous golden rectangle.



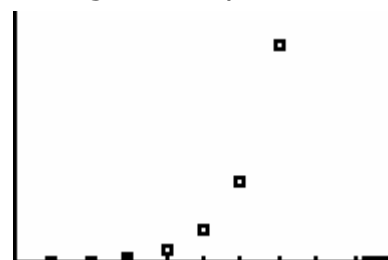
- Complete the table below to show the relationship between the number of a square in each golden rectangle and the area of the square. Let the 3×3 square be square #1.

Square Number	Area of Square
1	9
2	25
3	64
4	169
5	441
6	1156
7	3025

- Enter the square numbers into List 1 of a graphing calculator and the areas into List 2. Make a scatter plot for squares 1 – 7. Sketch your scatter plot below and describe the domain and range of the plot.

Domain $\{1, 2, 3, \dots, 7\}$

Range is between 9 and 3025



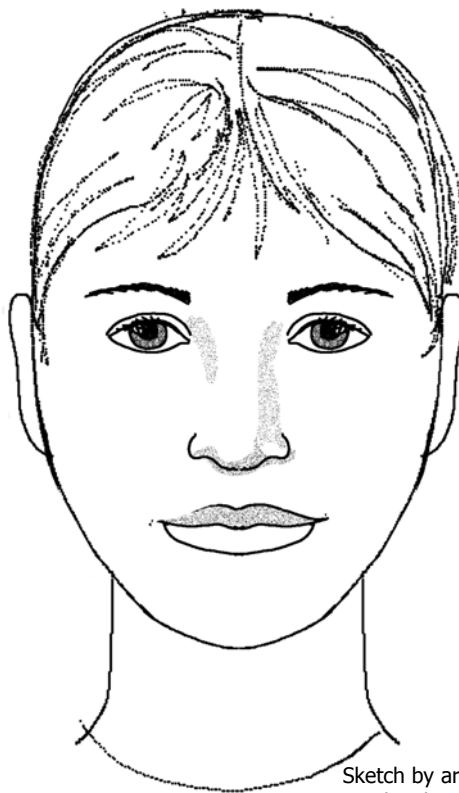
- Without using the regression feature on your calculator, write a function that fits your data. Enter your function into your calculator to test it. Alter the function as needed until you are satisfied that it fits the data.

$$y = 4(2.6)^x$$

- Explain how the numbers in your function are related to the data.
To get the first number, I backed up to the term before 3^2 to get 2^2 or 4 for the initial value. The base is the square of the golden ratio since we are using squares of the Fibonacci numbers.
- Would your function be any different if you started with 2^2 instead of 3^2 as the first area? If so, how and why?
I would have to change the coefficient to 1 because now 2^2 is the term that is paired with $x = 1$. The ratio would be still be 2.6 but it would not fit the points as well. The first three terms of the Fibonacci sequence don't approximate the golden ratio as well.

Transparency – The Eye of the Beholder

Throughout history and cultures, humans have been attracted to each other in various ways. One level of attraction has to do with a person's physical appearance – particularly the face. Mathematicians, artists, and physicians have studied certain features of the human face and determined that **ratios** of some measurements of features in the so-called "beautiful people" have a value very close to a specific number.

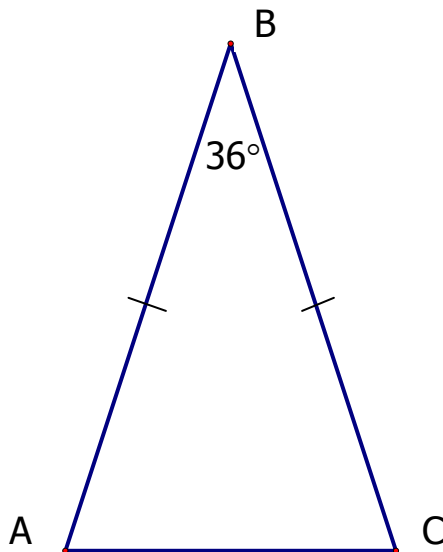


Sketch by artist, Debra L. Hayden, 2005.
Used with permission.

Artists use this so-called "**golden ratio**" to create images that are considered classically beautiful. Using Geometer's Sketchpad and your activity sheet, you will determine how close to "perfect" a celebrity of your choice seems to be.

Transparency 1 – The Golden Triangle

One geometric model of the **golden ratio** is an isosceles triangle with vertex angle of 36° .

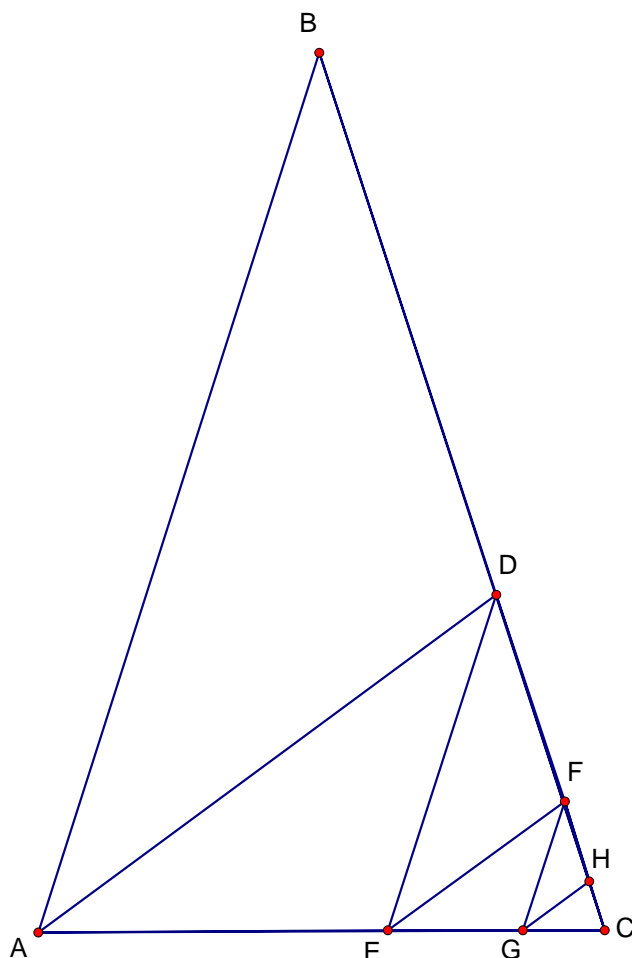


Bisect $\angle A$ and name the point where the bisector intersects \overline{BC} point D. What is true about $\triangle CAD$? How are $\triangle CAD$ and $\triangle ABC$ related?

Use Geometer's Sketchpad and the sketch **Golden Triangle** (Golden Triangle 1 tab) to determine the ratios $\frac{\overline{BC}}{\overline{AC}}$ and $\frac{\overline{BD}}{\overline{DC}}$.

Transparency 2 – The Golden Triangle

Repeat the process of bisecting a base angle several times. The results are shown here. See if you can identify the pairs of segments that fit the golden ratio.



Use the sketch **Golden Triangle** (Golden Triangle 2 tab) to find ratios for additional triangles. Record your results on the activity sheet **Creating a “Golden” Exponential Function**.

Transparency 3 – The Golden Section

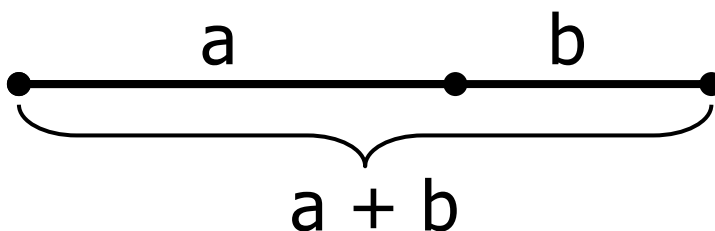
Consider the sequence of numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

After the first number, the ratios of any number to the preceding number eventually approximate what we call the “golden ratio”. The sequence of numbers was discovered by Leonardo Fibonacci around 1200 A.D.

Transparency 4 – The Golden Section

In geometry, if we take a segment and cut it to represent the golden ratio, it would look like this.



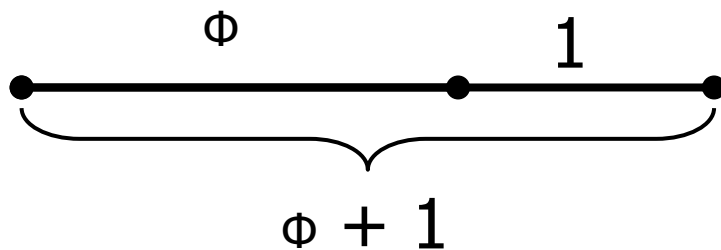
The ratio of the longer section to the shorter section is equal to the ratio of the whole segment to the longer section.

As a proportion, it looks like this:

$$\frac{a}{b} = \frac{a + b}{a}$$

Transparency 5: The *Algebra* of the Golden Ratio

Consider the proportion, $\frac{a}{b} = \frac{a+b}{a}$. The golden ratio is the value of $\frac{a}{b}$, but how can we find that value numerically? If we go back to the divided segment and start with the shorter section equaling 1, then the proportion becomes simpler to solve. We will also substitute the symbol, Φ (Greek letter phi), for the larger section.



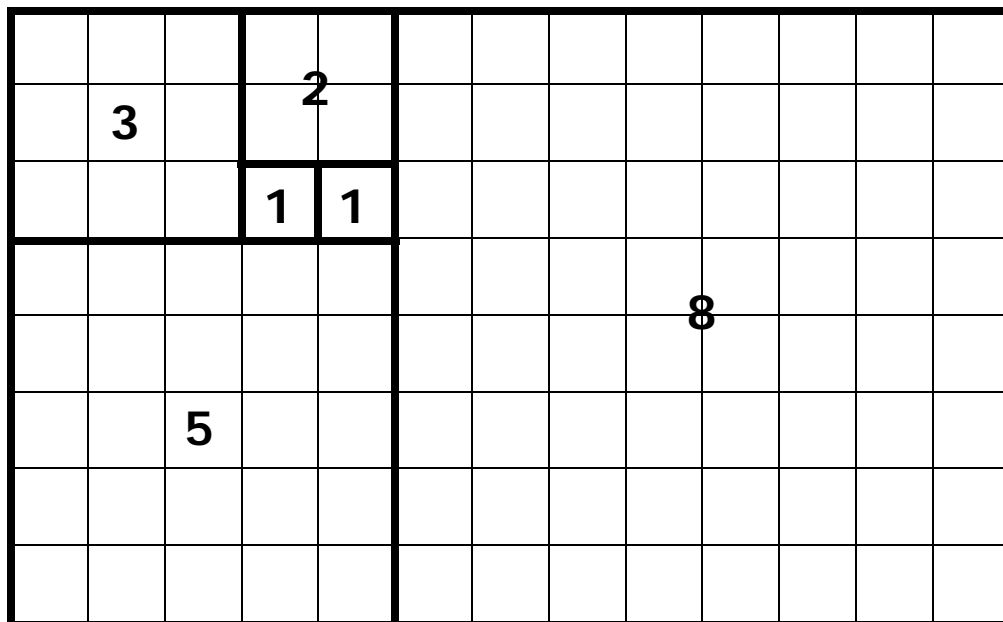
Transparency 6: The *Algebra* of the Golden Ratio

Now the proportion is $\frac{\Phi}{1} = \frac{\Phi + 1}{\Phi}$. Multiply the means and extremes to get $\Phi^2 = \Phi + 1$. The definition of the value of phi (Φ), the golden ratio, is a number whose value squared equals its value plus one.

Solve the quadratic equation $\Phi^2 = \Phi + 1$.

Transparency – The Golden Rectangle

If consecutive numbers from the Fibonacci sequence were the dimensions of a rectangle, we would have a “golden rectangle.” The unique ratios illustrated by a golden rectangle are said to be the most visually aesthetic of all ratios. People often use the numbers of the Fibonacci sequence to create a golden rectangle. Here’s one way to look at it. Each square has a Fibonacci number side length. Extend a side to create a length using the next Fibonacci number and you have a golden rectangle.

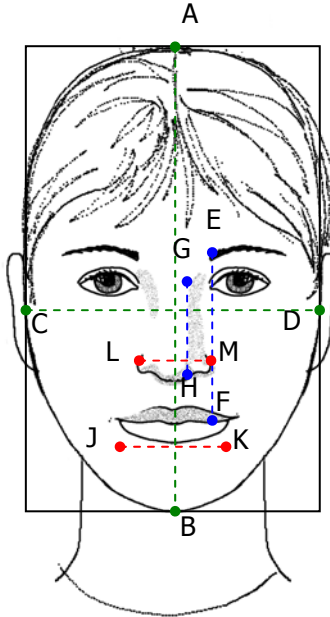


Can you see the rectangles with dimensions of 3×5 , 5×8 , and 8×13 ?

Look on the Internet for examples of how artists and architects have used the golden ratio in a rectangular format.

The Eye of the Beholder

- Study the features on the artist's sketch below. Identify the segments that represent each of the following ratios.



Sketch by artist, Debra L. Hayden, 2005.
Used by permission

Ratio	Segments
Length of face to Width of face	$\frac{\text{Length of face}}{\text{Width of face}} =$
Lips to eyebrows to Length of nose	$\frac{\text{Lips to eyebrows}}{\text{Length of nose}} =$
Width of mouth to Width of nose	$\frac{\text{Width of mouth}}{\text{Width of nose}} =$
Average Ratio	

- Open the sketch **Face Sample** in Geometer's Sketchpad. Calculate the ratio values indicated in the sketch by clicking on the "Measure Ratio" action button for the given ratio. Also, calculate the average ratio. Record your answers in your table.

In the Eye of the Beholder?
For each pair of facial measurements below, what is the length to width ratio?

Length of Face/
Width of face

Lips to eyebrows/
Length of nose

Width of mouth/
Width of nose

Golden
Ratio

- Log on to the Internet and open the website <http://www.angelfire.com/celeb2/celebrityfaces/>. Search for a photo of your favorite celebrity. The photo must be a full front view of the face.
- Right click on the face and select "Copy" so that you can "insert" the photo into Geometer's Sketchpad.
- Using Geometer's Sketchpad, construct and measure segments of the face you copied as shown on the sample. Measure the appropriate ratios and record them in the chart below.

I used a photo of :			
Length of face		Ratio	
Width of face			
Lips to eyebrows		Ratio	
Length of nose			
Width of mouth		Ratio	
Width of nose			

- How do your ratios compare with those found by other groups in the class? Why do you think this is so?

Creating a "Golden" Exponential Function

1. Open the sketch **golden triangle1** to find possible measurements for each of the following:

The length of \overline{BC} = _____

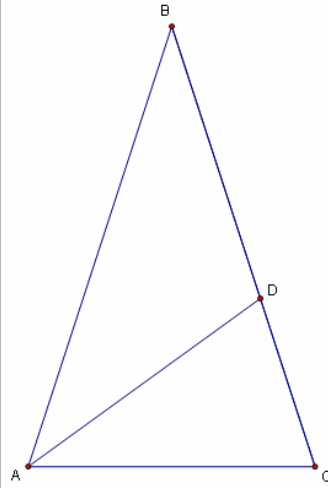
The length of \overline{AC} = _____

The ratio of $\overline{BC} : \overline{AC} \approx$ _____

The length of \overline{BD} = _____

The length of \overline{DC} = _____

The ratio of $\overline{BD} : \overline{DC} \approx$ _____



Use the Measure menu to find and display the lengths of segments BC, AC, BD, and DC.

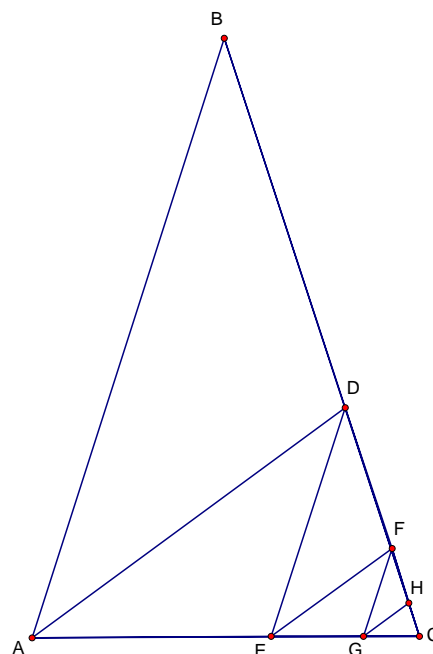
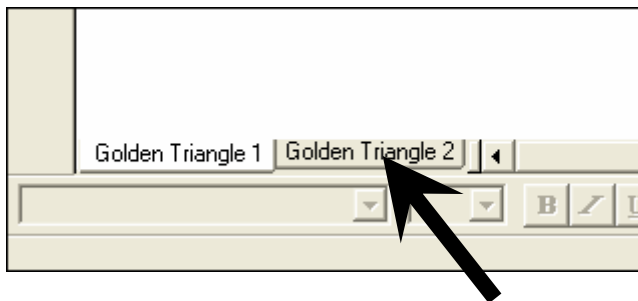
-
-
-
-

Measure and display the ratios:
 $\frac{BC}{AC}$ and $\frac{BD}{DC}$.

-
-

2. Click on Point C and drag it around the screen. What happens to the segment lengths?
3. What happens to the ratios when you drag point C around the screen?

4. Click the **Golden Triangle 2** tab inside your sketch. Find possible values for the following:



Triangle	Leg	Length	Successive Ratios
1	\overline{HC}		
2	\overline{GC}		
3	\overline{FC}		
4	\overline{EC}		
5	\overline{DC}		
6	\overline{AC}		
7	\overline{BC}		

5. Enter the numbers 1 – 7 into List 1 on your graphing calculator. Enter the lengths of the segments \overline{HC} , \overline{GC} , \overline{FC} , \overline{EC} , \overline{DC} , \overline{AC} , and \overline{BC} into List 2. Create a scatter plot on your graphing calculator with the Triangle number on the x -axis and the Leg length on the y -axis. Sketch your plot and describe your window.

Algebra and the Golden Ratio

You have found the exact value of the golden ratio to be $\frac{1+\sqrt{5}}{2}$. Let's look at how this value connects to the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...

Consider the table below but don't fill in the right-hand column until you've answered questions 1 – 3.

Term number	Fibonacci number
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

1. If you made a scatter plot of Fibonacci number vs. term number, what would the scatter plot look like?
2. If you started with 1 as your first Fibonacci number, could you write a function that would pass through all of the points in your scatter plot?
3. How could you make a scatter plot that more closely fits an exponential function?
4. Fill in the table with the Fibonacci numbers of your choice and write an exponential function to fit your points.
5. Which would give a better fit: starting with 5 or starting with 13? How does choosing a different starting number affect your function rule?

The Golden Ratio in Art and Architecture

Search the Internet using key words "golden ratio" and "art" or "architecture." Find one example of how the golden ratio is used in art and one example of its use in architecture. Record at least the following information for each example.

Art Example

The artist is/was _____

The name of the painting, sculpture, etc. is _____

Give a brief description or simple sketch of how the golden ratio is used in this work.

Architecture Example

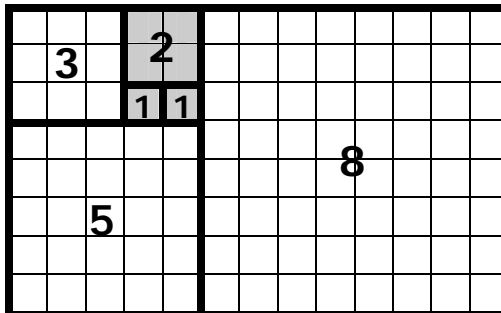
The architect is/was _____(or give country where it is located)

The name of the painting, sculpture, etc. is _____

Give a brief description or simple sketch of how the golden ratio is used in this structure.

Golden Areas

Consider the squares that make up a golden rectangle shown below. The squares have sides that are terms of the Fibonacci sequence: 1, 1, 2, 3, 5,... Each golden rectangle, such as the square that is shaded, is formed by attaching the next Fibonacci square to the previous golden rectangle.



- Complete the table below to show the relationship between the number of a square in each golden rectangle and the area of the square. Let the 3×3 square be square #1.

Square Number	Area of Square
1	3^2
2	5^2
3	
4	
5	
6	
7	

- Enter the square numbers into List 1 of a graphing calculator and the areas into List 2. Make a scatter plot for squares 1 – 7. Sketch your scatter plot below and describe the domain and range of the plot.



- Without using the regression feature on your calculator, write a function that fits your data. Enter your function into your calculator to test it. Alter the function as needed until you are satisfied that it fits the data.
- Explain how the numbers in your function are related to the data.
- Would your function be any different if you started with 2^2 instead of 3^2 as the first area? If so, how and why?

- 1 A stationery company makes cards and posters using dimensions of golden rectangles. So far their inventory includes posters with dimensions (in inches) of 3×5 , 5×8 , and 8×13 . Which equation below would be useful in approximating the length of a poster with a width of 21 inches?

- A $L = 13 \times 21$
- B $L = 3 \times 1.6^4$
- C $L = 13 + 21$
- D $L = (1.6)(21)$

- 2 The table below shows a section of the Fibonacci sequence.

Term number x	Fibonacci number y
0	5
1	8
2	13
3	21
:	:

Which function best fits the data shown in the table?

- A $y = 1.6x$
- B $y = 5 * 1.6^x$
- C $y = x^{1.6}$
- D $y = 8 * 1.6^x$

- 3 The exact value of ϕ , referred to as the golden ratio, can be found by taking the larger root of the equation $x^2 = x + 1$. What is the exact value of ϕ ?

- A $\frac{5}{3}$
- B 1.618
- C $\frac{1 + \sqrt{5}}{2}$
- D $\frac{1 - \sqrt{5}}{2}$

- 4 The function $y = 2(1.62)^x$ produces the table below when the domain is $\{1, 2, 3, \dots\}$.

X	Y1	
1	3.24	
2	5.2488	
3	8.5031	
4	13.775	
5	22.315	
6	36.151	
7	58.565	

X=1

Which function will produce the table

X	Y1	
1	8.5031	
2	13.775	
3	22.315	
4	36.151	
5	58.565	
6	94.875	
7	153.7	

X=1

for the same domain?

- A $y = 1.2346 * 1.62^x$
- B $y = 3.24 * 1.62^x$
- C $y = 5.2488 * 1.62^x$
- D $y = 8.5031 * 1.62^x$